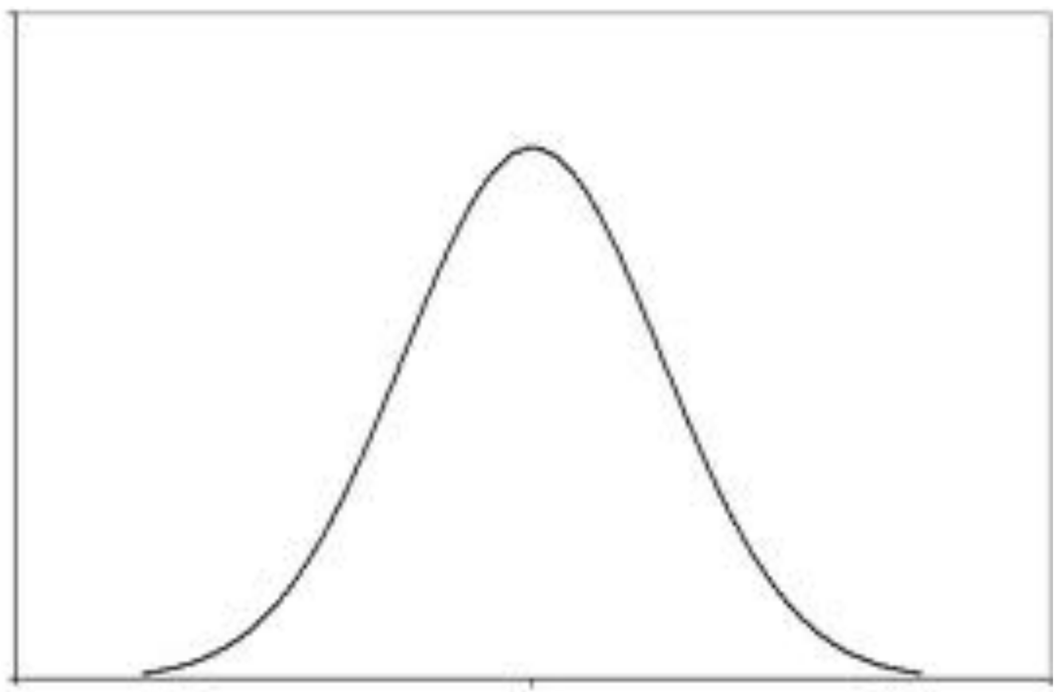
Matching Basics

Which matching?

- The Nobel classification:
  - **Roth-Shapley**
    - How to get to a stable match via centralized mechanisms
    - General preferences
  - **Diamond-Mortensen-Pissarides**
    - How certain decentralized mechanisms (markets with search frictions) generate assignments; typically more structure on preferences/technology
  - **Becker**
    - Frictionless benchmark: characterization of stable matches with structure on preferences/technology

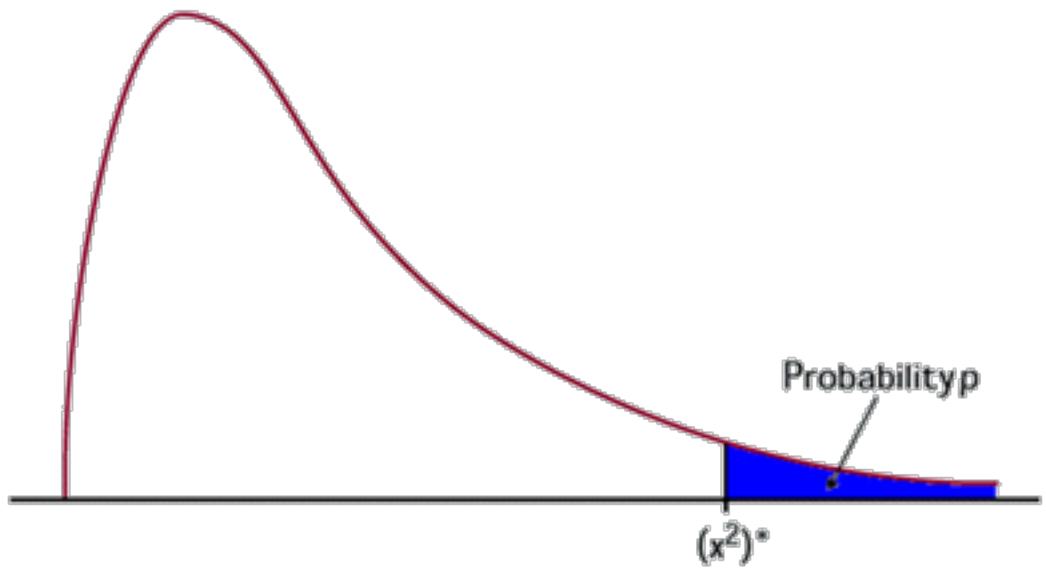
Old puzzle

- Typical observed distribution (height or IQ or task completion time)



Old puzzle

- Typical earnings distribution



Enter the matching model

- (Roy 1950; Tinbergen 1951)
- One worker's productivity depends on the resources at one's disposal (especially other workers)
- Key insight: **complementaries** will lead to **assortative mathing** (high skill with high skill, low with low)
- Result: distribution of earnings *may* be skewed even if skill distribution is symmetric

The argument

- Suppose that firms consist (for simplicity) of two workers
- Output is $h(a,b)$ when a worker of skill *a* is paired with a worker of skill *b*
  $h_1 > 0, h_2, h_{12} > 0$
- Skill is distributed according to some symmetric distribution (say, normal)
- A competitive equilibrium (efficient allocation) will have assortative matching (to be shown)

The argument, cont.

- Thus, type *a* receives a wage equal to $\frac{1}{2}h(a,a)$
- Say $h(a,b) = ab$; then wage for *a* is

$$w(a) = \frac{1}{2}a^2$$

- If *a* is distributed normally, then *w* has a chi square distribution, hence skewed.

- Q: what happens if $h(a,b) = \sqrt{ab}$? $(ab)^{1/3}$?

Outline

- Matching Basics: solution concept, matching patte
- patterns, TU vs NTU
- TU: conditions leading to nice and not-so-nice matching patterns
- Strict NTU
- Everything in between: NTU
- Endogenous types, segregation, and policy
- Working women and divorce (Friday at 12)

Environment

- Economy consists of a large number of individuals $i \in I$ (I can be finite, countable or a continuum) distinguished by "types" $t \in T = A \times G$, where $A \subset \mathbb{R}$ is compact, and

$$G = \begin{cases} \{0,1\} & \text{"Two-sided" model} \\ \{0\} & \text{"One-sided" model} \end{cases}$$

- Type assignment function $\tau : I \to T$
- For every subset of the population ("coalition") there is a payoff possibility set (a subset of $\mathbb{R}^n_+$ for coalition of n people)
- In a *size-k matching model* this set depend only on the types of the coalition's members *and* the PPS for coalitions larger than k is just the union of PPS's for coalitions of size up to k (no externalities across coalitions). Usually (as here) $k = 2$
- Singletons get 0 as a simplification.
- In the two-sided model, $V((a,0),(a',0)) = V((b,1),(b',1)) = \{(0,0)\}$ for all $a, a', b, b' \in A$

Transferability

- *transferable utility* (TU) if every PPS has the form

$$V(t,t') = \{(u_1,u_2) \in \mathbb{R}^2_+ | u_1 + u_2 \leq h(t,t')\},$$

  (i.e., the Pareto frontier is the "off-diagonal")
- *non-transferable utility* (NTU) if it isn't TU. We will consider the case in which

$$V(t,t') = \{(u_1,u_2) \in \mathbb{R}^2_+ | u_2 \leq \phi(t',t,u_1)\},$$

  where $\phi(t',t,u_1)$ is strictly decreasing in u_1
- TU is the special case that $\phi(t',t,u) = h(t',t) - u$
- Limiting case: *strictly non-transferable utility* (SNTU) if every PPS has the form

$$V(t,t') = \{(u_1,u_2) \in \mathbb{R}^2_+ | u_1 \leq u(t,t') \ \&\ u_2 \leq u(t',t)\}$$

  (i.e. the strict Pareto frontier is a point)

Stable Matches

- A *stable match* (M^*, U^*) consists of a one-to-one matching function $m^* : I \to I$ and a payoff allocation $U^* : I \to \mathbb{R}_+$ satisfying
  1. Feasibility: $(U^*(i), U^*(M^*(i)) \in V(\tau(i), \tau(M^*(i))$ whenever $M^*(i) \neq i$, else $U^*(i) = 0$
  2. Stability: $U^*(i) \geq 0$, all i, and there are no i, j and $(u_1, u_2) \in V(\tau(i), \tau(j))$ such that $(u_1, u_2) > (U^*(i), U^*(j))$
- Stability is sometimes referred to as the *no-blocking* condition
- $M^*(i) = i$ is interpreted as autarky for i
- Equivalent to the core and the f-core; sometimes stable match is called "equilibrium"

Immediate implications of match stability

- The stability condition implies $(U^*(M^*(i)), U^*(i))$ is on the Pareto frontier $\phi(\tau(M^*(i)), \tau(i), U^*(i))$ of $V(\tau(i), \tau(M^*(i)))$
- Indeed, the stable match is ("constrained") Pareto optimal
- Whenever the frontier is strictly decreasing (all our cases except SNTU) there is *equal treatment*: if $\tau(i) = \tau(j)$ then $U^*(i) = U^*(j)$; thus we will write $u^*(t)$ for $U^*(i)$, where $t = \tau(i)$, and $m * (t) = \tau(M^*(\tau^{-1}(t)))$ (in general m^* is a correspondence)
- The stability condition can be written

$$\phi(t, m^*(t), u^*(m^*(t))) \geq \phi(t, t', u^*(t')), \text{all } t'$$

- This optimality property is useful for computing equilibrium payoffs (see?) but we need more results before we can get there.
- In two-sided models, stable matches are only "across sides"

Some Monotone Matching Patterns ("Matterns") I

- There is *segregation* (SEG) if $m^*(t) = t$, all t
  - Strictly speaking, this makes sense only in one-sided models; sometimes people say a two-sided match is segregated if $m^*((0,a)) = (1,a)$, all a
- There is *positive assortative matching* (PAM) if for any two matched types $\langle t, t' \rangle$ and $\langle s, s' \rangle$, we have

$$max\{t, t'\} > max\{s, s'\} \Longrightarrow min\{t, t'\} \geq min\{s, s'\}$$

  - By lexicographically ordering types first by gender, then by attribute, this yields the intuitive definition for two-sided models, e.g., more talented workers are matched to more productive firms.
  - SEG is a form of PAM; but in general PAM allows for heterogenous matches

Some Monotone Matching Patterns ("Matterns") II

- Another form of PAM for one sided models: construct an attribute distribution from the type assignment τ; call its c.d.f. F and let a_m be the median attribute; if F is continuous on $A = [\underline{a}, \overline{a}]$ then we can define *median matching* (MED) by

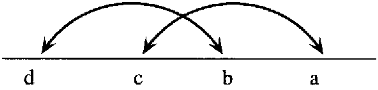
$$F(a) - F(m^*(a)) = \frac{1}{2}, \text{ all } a \in [a_m, \overline{a}]$$

- There is *negative assortative matching* (NAM) if for any two matched types $\langle t, t' \rangle$ and $\langle s, s' \rangle$, we have

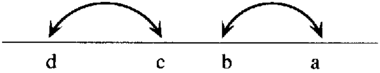
$$max\{t, t'\} > max\{s, s'\} \Longrightarrow min\{t, t'\} \leq min\{s, s'\}$$

- With a continuous c.d.f. NAM gives

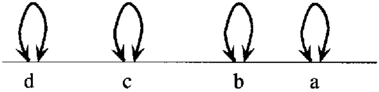
$$F(a) + F(m^*(a)) = 1, \text{ all } a \in [\underline{a}, \overline{a}]$$



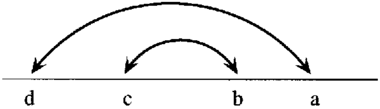
PAM
not NAM
possibly median matching
not segregation



PAM
not NAM
not median matching
not segregation



PAM
not NAM
not median matching
segregation



not PAM
NAM
not median matching
not segregation

Transferable Utility

TU Preliminaries

- **Lemma** Under TU, the stable match m^* maximizes the aggregate payoff.
  Proof If m^* fails to maximize the aggregate payoff, there is a matching correspondence $\hat{m}$ and pair of types t and t' with $t' \in \hat{m}(t)$ and $t' \notin m^*(t)$ such that $h(t,t') > u^*(t) + u^*(t')$. But then $\langle t, t' \rangle$ would have blocked m^*, a contradiction.
- The *segregation payoff* for type t is

$$\underline{u}(t) \equiv h(t,t)/2.$$

- The *surplus* for a pair of types t and t' is

$$\sigma(t,t') = max\{0, h(t,t') - \underline{u}(t) - \underline{u}(t')\}.$$

Conditions leading to various matterns

- Assume one-side, h is symmetric ($h(a, b) = h(b, a)$), and there is an even number or continuum of every type until further notice
- A function $h : \mathbb{R}^2 \to \mathbb{R}$ satisfies *increasing differences* (ID) if

$$a > a', b > b' \implies h(a, b) - h(a, b') \geq (a', b) - h(a', b')$$

  - Equivalent to supermodularity on $\mathbb{R}^2$; if h is smooth then $\frac{\partial^2 h}{\partial a \partial b} > 0$
- **Proposition** If h satisfies strict ID (i.e. $>$ replacing $\geq$ everywhere), stable matches are segregated.
  Proof If $m^*(a) \ni b > a$, then since by strict ID $h(b, b) - h(b, a) > h(a, b) - h(a, a)$, we have $h(b, b) + h(a, a) > 2h(a, b)$, violating aggregate payoff maximization.
- A weaker condition leading to (payoff equivalence to) segregation is that $\sigma(a, b) = 0$ for all a, b.

Heterogeneous Matterns

- If instead $\sigma(a, b) > 0$ for all $a \neq b$ ("σ is positive off the diagonal"), then matches must be heterogeneous
- **Example** The *post-match task assignment* model (Kremer-Maskin 1996) has

$$KM(a, b) \equiv max\{a^{\theta} b^{1-\theta}, b^{\theta} a^{1-\theta}\}, \text{for } \theta \in (\frac{1}{2}, 1). \tag{1}$$

  This is not everywhere differentiable, though its cross-partial is positive wherever it is defined. It doesn't satisfy ID there since for $a > b$ we would need $a + b > 2a^{\theta} b^{1-\theta}$, which fails for b close enough to a.
- Thus if the support of F is convex, matching is heterogenous for this function.

WID

- A function $h : \mathbb{R}^2 \to \mathbb{R}$ satisfies *weak increasing differences* if for any $a > b \geq c > d$ we have

$$h(b,c) - h(b,d) \leq h(a,c) - h(a,d) \textbf{ or}$$
$$h(b,d) - h(c,d) \leq h(a,b) - h(a,c)$$

- **Proposition** If h satisfies WID, stable matches are payoff equivalent to PAM.
  Proof PAM $\Leftrightarrow$ Nowhere is there a quadruple $a > b \geq c > d$ with $\langle a, d\rangle$ and $\langle b, c\rangle$ matched. If there were, then $h(a,d) + h(b,c) > h(a,c) + h(b,d)$ and $h(a,d) + h(b,c) > h(a,b) + h(c,d)$ (strictness applies, else a PAM arrangement could generate the same payoffs). But this violates WID.
- Since for $a > b \geq c > d$, $a^{\theta}(c^{1-\theta} - d^{1-\theta}) > b^{\theta}(c^{1-\theta} - d^{1-\theta})$, the function KM generates PAM.

How heterogeneous?

- A strengthening of WID : h satisfies "Condition M" if if for any $a > b \geq c > d$ we have

$$h(b, c) - h(b, d) \leq h(a, c) - h(a, d) \textbf{ and}$$
$$h(b, c) - h(c, d) \leq h(a, b) - h(a, d)$$

  - This strengthens WID but not ID because ID requires the second inequality to be reversed.
- **Proposition** If σ is positive off the diagonal and Condition M is satisfied, then there is MED.

- *KM* **and segregation in firms**. If $A = [\underline{a}, \overline{a}] \subset \mathbb{R}_{++}$, then M is satisfied if $\underline{a} > \left(\frac{1-\theta}{\theta}\right)^{1/\theta} \overline{a}$.
  - For a "tight" skill distribution, there is median matching, which minimizes numerical measures of segregation; as the skill distribution "stretches", we lose median matching, which increases the measure of segregation.
  - If $\theta = \frac{1}{2}$, there is segregation. As θ grows (skill-biased technical change?), matching becomes heterogeneous, with minimal segregation eventually reached as condition M become satisfied.

NAM

- A sufficient condition for NAM is that h satisfy *decreasing differences* (DD)

$$a > a', b > b' \Longrightarrow h(a,b) - h(a,b') \leq (a',b) - h(a',b')$$

- A weaker sufficient condition is $\sigma(a,b) > 0$ when $a \neq b$ and *weak decreasing differences* (WDD):

$$\begin{aligned} a > b \geq c > d \Longrightarrow h(b,c) - h(b,d) &\geq h(a,c) - h(a,d) \textbf{ and} \\ h(b,c) - h(c,d) &\geq h(a,b) - h(a,d) \end{aligned}$$

Necessary Conditions – the KIS principle

- The results so far are sufficient for monotone matching. They are also *distribution free*. That is, if, say, WID is satisfied on an interval $I \subset A$, the match will have the PAM property for any distribution of types with support in I.
- So if a necessary condition fails, there exists a type distribution for which the stable match fails to have the desired property.
- The surplus function σ inherits the difference properties of its payoff function h. But not vice versa! So necessary conditions are expressed in terms of the surplus.

Necessary Conditions – the Keep It in terms of Surplus principle

- **Proposition** $\sigma \equiv 0$ is necessary as well as sufficient for segregation.
- **Proposition** A necessary as well as sufficient condition for PAM is that σ satisfies WID for all "test quadruples" $a > b \geq c > d$ with $\sigma(a, d) > 0$.
- **Proposition** A necessary as well as sufficient condition for NAM is that σ is positive off the diagonal and satisfies WDD.
- **Remark** In two sided-models, the payoff and surplus coincide, so these conditions can be expressed in terms of payoff in that case.

Modeling Imperfections I I

- Many imperfections will introduce some non-transferabilty into the model. But not all.
- **Example** Production requires a capital investment $k > 0$. All agents have zero wealth and skill $a \in [\underline{a}, \overline{a}]$, where $\underline{a} > \sqrt{k}$. There is an imperfect credit market: the severity of the imperfection is indexed by $\phi \geq 1$, where 1 is a perfect market. Specifically,

$$h(a, a') = \begin{cases} aa' - k & \text{if } aa' \geq \phi k \\ 0 & \text{if } aa' < \phi k \end{cases}$$

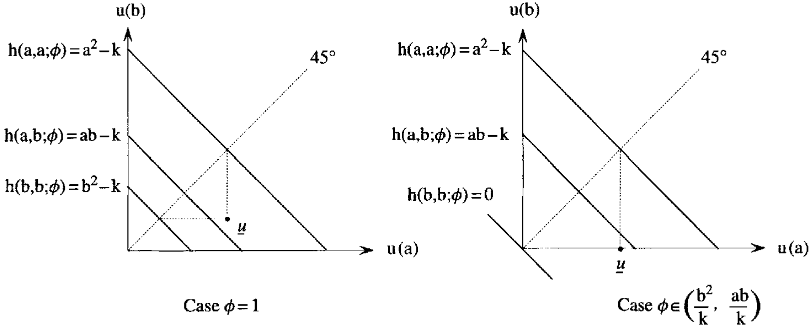
  - Derive this from a "running off with the till" model with $\phi = \frac{1}{1-\epsilon}$, where ϵ is the escape probabilty.
- With $\phi = 1$, there is SEG, since h satisfies ID on $[\underline{a}, \overline{a}]$. Note all types can generate positive payoffs

Modeling Imperfections I II

- With $\phi > 1$, the segregation payoff drops to 0 for the lowest types (those smaller than $\sqrt{\phi k}$); higher types' segregation payoffs are unchanged.

Modeling Imperfections II

- Suppose $b < \sqrt{\phi k} < a$ with $ab > \phi k$. If $\sigma(a, b) = ab - k/2 - a^2/2 > 0$, then a matching is now heterogeneous.



- One can in fact obtain NAM for certain type distributions, non-monotone patterns for others.

SNTU

- Strict NTU does not require difference conditions to obtain monotone matching (Becker, 1973)
- It is enough that payoffs are monotone in partner's type:
  - **Proposition** If $b > b' \implies u(a, b) > u(a, b')$, all a, there is segregation (PAM in two-sided models).
  - **Proof** Suppose $a > b$ and there are two matches of the form $\langle a, b \rangle$. Since $u(a, a) > u(a, b)$, $\langle a, a \rangle$ would block, a contradiction.

Everything in between: Nontransferabilities

2 Examples

2.1 Example I: Risk Sharing in Households

- marriage market in which the primary desideratum in choosing a mate is suitability for risk sharing

- denote by p the characteristics of the men and by a the characteristics of the women

- household production is random, with two possible outcomes $w_2 > w_1 > 0$, and associated probabilities π_2 and π_1

- income y yields utility $U(p, y)$ to a man of type p and $V(a, y)$ to a woman of type a

- U and V are twice differentiable, strictly increasing and strictly concave in income for all p and a

- characteristics p and a are indices of absolute of risk tolerance:

  - if $p > p'$, then $-U_{22}(p, y)/U_2(p, y) < -U_{22}(p', y)/U_2(p', y)$ for all y

  - $a > a'$ implies $-V_{22}(a, y)/V_2(a, y) < -V_{22}(a', y)/V_2(a', y)$.

- frontier for a man of type p who is matched to a woman of type a is given by solution to the optimal risk sharing problem:

$$\phi(p,a,v) \equiv \max_{\{s_i\}_{i=1,2}} \Sigma_i \pi_i U(p, w_i - s_i) \text{ s.t. } \Sigma_i \pi_i V(a, s_i) \geq v.$$

- The question is: which type of women match to each type of man?

- Since ϕ is generally not linear in v, utility is only imperfectly transferable: the cost to p of transferring a small amount to a depends on how much each partner already has

2.2 Matching Principals and Agents

- principals' projects have a common expected return but differ in their risk characteristics; agents, who differ in initial wealth

- principal's type $p \in (0, 1]$ indexes the success yield and probability of his project:
  - it yields R/p with probability p and 0 with probability $1 - p$ provided his agent exerts $e = 1$;
  - it yields 0 with probability 1 if $e = 0$
  - common expected return, but riskiness declines with p

- agents type index $a > 0$ represents initial wealth
  - type a has utility $V(a+y)$ from income y
  - $V' > 0 > V''$, and it displays increasing absolute risk tolerance.

- The frontier for a principal of type p who is matched to an agent of type a is given by

$$\phi(p,a,v) = \max \; R - ps_1 - (1-p)s_0$$

$$\text{s.t. } pV(a+s_1) + (1-p)V(a+s_0) - 1 \geq V(a+s_0), \quad \text{(Incentive Compatibility)}$$

$$pV(a+s_1) + (1-p)V(a+s_0) - 1 \geq v \quad \text{(Individual Rationality)}$$

where s_1 and s_0 are the wages paid in case of success and failure respectively.

- Intuition: since wealthier agents are less risk averse, they should be matched to riskier tasks while the more risk averse agents should accept the safer tasks (i.e. there should be *negative assortative matching* in (p, a))

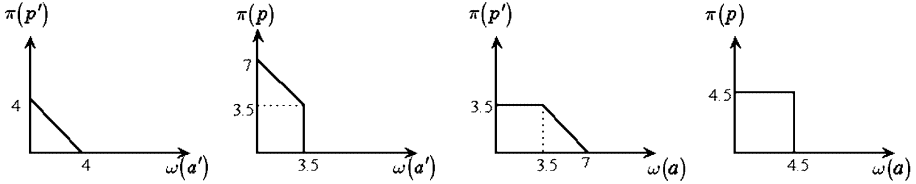
New effects of NTU

- Suppose "output" depends on types as follows:

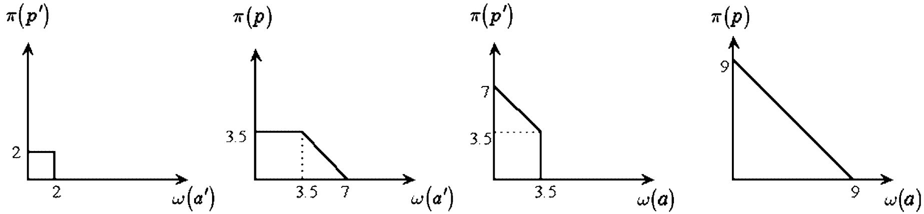
	p	p'
a	9	7
a'	7	4

- Under TU, the match is NAM $\langle p, a' \rangle, \langle p', a \rangle$
- Under equal sharing (SNTU), the match is PAM $\langle p, a \rangle, \langle p', a' \rangle$

- What about intermediate cases?



(1) Transferability decreasing in type



(2) Transferability increasing in type

FIGURE 2.1.—Utility possibility frontiers.

3 Model

- Two "sides" of agents with compact attribute spaces P and $A \subset \mathbb{R}$
- number of agents is finite
- Utility possibility frontier $\phi(p, a, v)$: utility to p in a match with a when a receives v
- ϕ is continuous, non-decreasing in (p, a) and decreasing in v
- Similarly, define the "inverse" $\psi(a, p, u) : \phi(p, a, \psi(a, p, u)) = u$
- TU $\phi(p, a, v) = h(p, a) - v$

3.1 Equilibrium Matching Patterns

- Use core (stable matching):
  - $\pi^* : P \to \mathbb{R}$ and $\omega^* : A \to \mathbb{R}$ be equilibrium payoffs
  - $\mathcal{M}^* : P \to A$ matching correspondence
  - stability: there is no p, a and $v > \omega^*(a)$ such that $\phi(p, a, v) > \pi^*(p)$

- Matching is *monotone* if $\mathcal{M}$ is a monotone correspondence:
  - *Positive assortative matching (PAM)* if for any two matched pairs (p, a) and $(\hat{p}, \hat{a})$, $p > \hat{p}$ implies $a \geq \hat{a}$
  - *Negative assortative matching (NAM)* if for any two matched pairs (p, a) and $(\hat{p}, \hat{a})$, $p > \hat{p}$ implies $a \leq \hat{a}$

3.2 Logic of the TU case

- If $h(\cdot, \cdot)$ satisfies *increasing differences* (ID):

$$p > p' \text{ and } a > a' \Longrightarrow h(p, a) - h(p', a) \geq h(p, a') - h(p', a')$$

  then the equilibrium match is PAM for any type distribution

- Suppose not, i.e. $p > p'$ and $a > a'$ with $\langle p, a' \rangle$ and $\langle p', a \rangle$ equilibrium matches not payoff equivalent to PAM

- then $\omega^*(a') > h(p', a') - \pi(p')$, so

$$\begin{aligned}
\pi^*(p) &= h(p, a') - \omega^*(a') \\
&< h(p, a') - [h(p', a') - \pi^*(p')] \quad \text{(by stability)} \\
&\leq h(p, a) - [h(p', a) - \pi^*(p')] \quad \text{(by ID)} \\
&= h(p, a) - \omega^*(a)
\end{aligned}$$

- thus $\pi^*(p) + \omega^*(a) < h(p, a)$ and $\langle p, a \rangle$ will block the match, a contradiction

- ID ensures that a can always (i.e. for any $\pi^*(p'))$ outbid a' for a higher partner

- Our generalized conditions for NTU will impose this same requirement on the frontier functions

4 Generalized Difference Conditions

- Definition:

  - GID: $p > p'$, $a > a'$, and $u \in [0, \phi(p', a', 0)] \Rightarrow \phi(p, a, \psi(a, p', u)) \geq \phi(p, a', \psi(a', p', u))$

  - GDD: $p > p'$, $a > a'$, and $u \in [0, \phi(p', a', 0)] \Rightarrow \phi(p, a, \psi(a, p', u)) \leq \phi(p, a', \psi(a', p', u))$

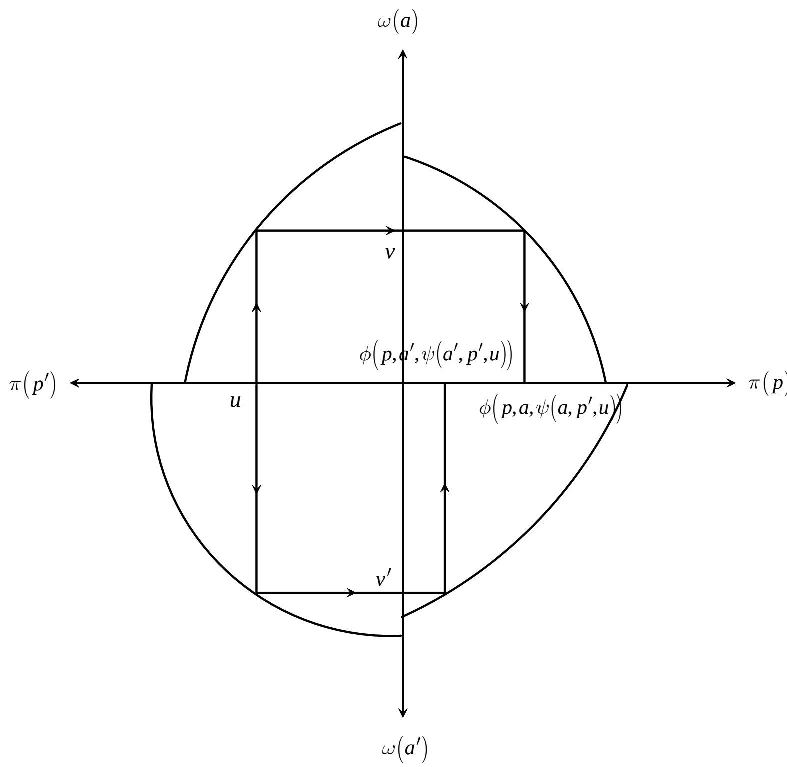
- Proposition 1: (i) GID implies payoff equivalence to PAM for all type distributions; (ii) GDD implies payoff equivalence to NAM.

- Proof for strict case: Say (p, a') and (p', a) are part of a stable match; stability implies

$\omega^* (a') \geq \psi (a', p', \pi^* (p'))$
and $\pi^* (p) \geq \phi (p, a, \omega^* (a))$;
but $\phi (p, a, \omega^* (a)) = \phi(p, a, \psi(a, p', \pi^* (p'))) > \phi(p, a', \psi(a', p', \pi^*(p')))$
(strict GID)
$\geq \phi(p, a', \omega^*(a'))$; (payoff monotonicity)
thus $\phi (p, a, \omega^* (a)) > \pi^*(p)$, contradicting stability

- Remark: if GID holds weakly there may be several equilibria, but there is always one with PAM and the others generate the same payoffs

4.1 Understanding the GDCs



- So GID

  can be written: $\phi(p', a, v) = \phi(p', a', v') \Longrightarrow \phi(p, a, v) \geq \phi(p, a', v')$

4.2 Example: Risk Sharing

- We claim that GDD is satisfied (strictly) by the risk sharing model. We verify that

$$\phi\left(p',a',v'\right)=\phi\left(p',a,v\right)\Rightarrow\phi\left(p,a,v\right)<\phi\left(p,a',v'\right),$$

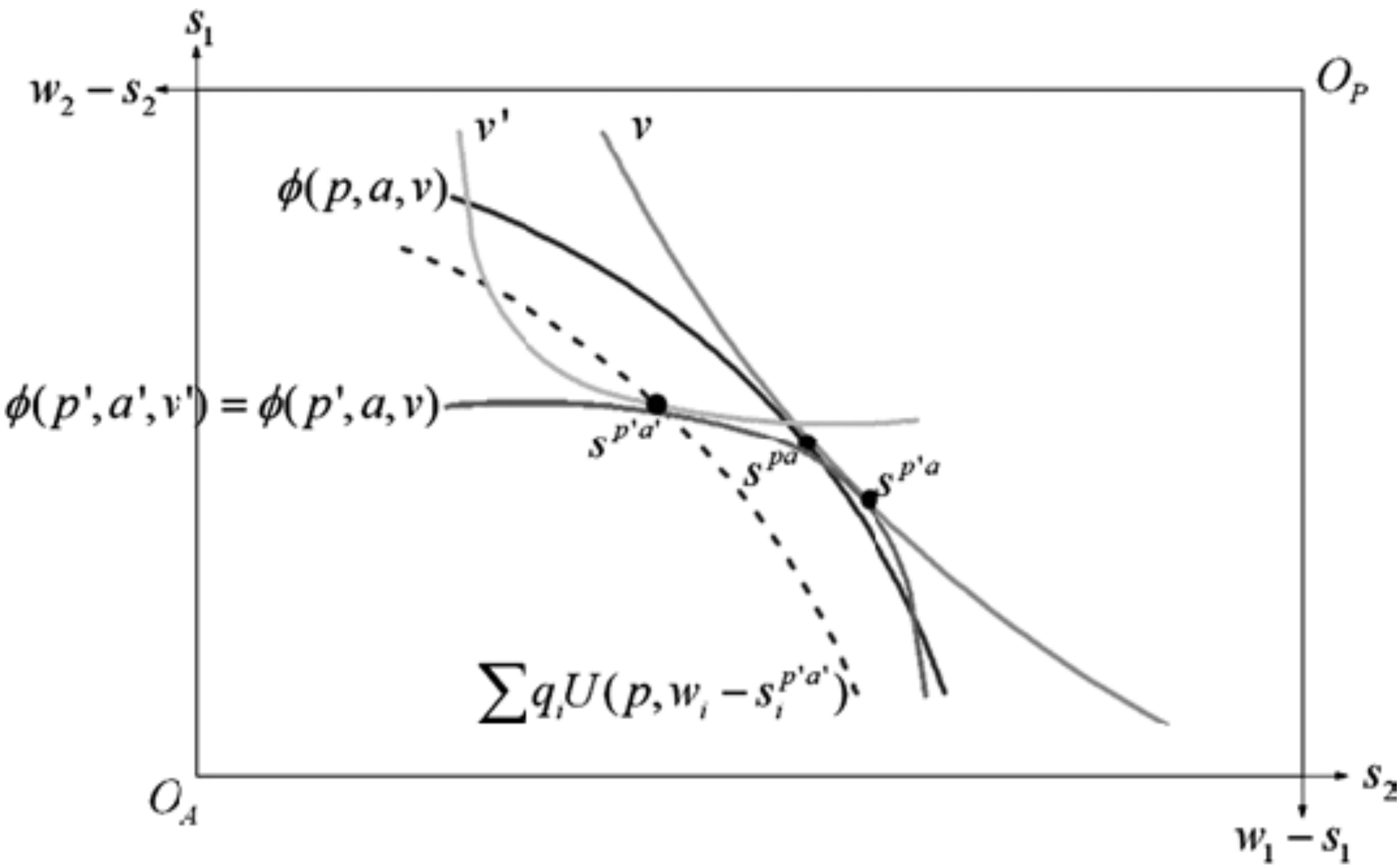
where

$$\phi(p,a,v)\equiv\max_{\{s_i\}}\Sigma_i\pi_i U(p,w_i-s_i)\text{ s.t. }\Sigma_i\pi_i V(a,s_i)\geq v \quad (1)$$

$$\phi(p,a',v')\equiv\max_{\{s_i\}}\Sigma_i\pi_i U(p,w_i-s_i)\text{ s.t. }\Sigma_i\pi_i V(a',s_i)\geq v', \quad (2)$$

etc., and v and v' are such that $\phi\left(p',a',v'\right)=\phi\left(p',a,v\right)$.

- Call the solutions s^{pa} for (1), $s^{pa'}$ for (2), etc., These are illustrated in the figure.



s_1
$w_2 - s_2$
O_P
v'
v
$\phi(p,a,v)$
$\phi(p',a',v') = \phi(p',a,v)$
$s^{p'a'}$
s^{pa}
$s^{p'a}$
$\sum q_i U(p, w_i - s_i^{p'a'})$
O_A
s_2
$w_1 - s_1$

- Showing GDD amounts to showing that

$$\Sigma_i \pi_i U(p, w_i - s_i^{pa'}) \geq \Sigma_i \pi_i U(p, w_i - s_i^{pa}).$$

- But it's "obvious" that

$$\Sigma_i \pi_i U(p, w_i - s_i^{p'a'}) > \Sigma_i \pi_i U(p, w_i - s_i^{pa}),$$

  from which it follows that $\Sigma_i \pi_i U(p, w_i - s_i^{pa'}) > \Sigma_i \pi_i U(p, w_i - s_i^{pa})$, again by revealed preference.

- Thus GDD is strictly satisfied, and we conclude that *in the risk-sharing economy men and women will always match negatively in risk attitude*.

4.3 Necessity

- GID / GDD are necessary for monotone matching:

- Proposition 2: If the outcome is PAM for all type distributions, the frontier satisfies GID

  – if instead there are $p > p'$, $a > a'$, and u such that $\phi\left(p, a, \psi(a, p', u)\right) < \phi\left(p, a', \psi(a', p', u)\right)$, then there is a distribution of types for which there is an equilibrium that is not payoff equivalent to PAM

  – for instance with equal masses at p, p', a, and a' the matches $\langle p', a\rangle$ and $\langle p, a'\rangle$ with payoffs and $(u, \psi(a, p', u))$ and $(\phi(p, a', \psi(a', p', u) + \epsilon), \psi(a', p', u) + \epsilon)$ is stable

- GID / GDD are ordinal conditions, invariant to increasing transformations of the payoff functions

5 Sufficient Differential Conditions

- Assume that frontiers are increasing in types and are twice differentiable

- Corollary 2:
  (i) a sufficient condition for PAM is that for all $(p,a) \in P \times A$ and $v \in (0, \psi(a,p,0))$,

$$\phi_{12}(p,a,v) \geq 0 \text{ and } \phi_{13}(p,a,v) \geq 0$$

  (ii) a sufficient condition for NAM is that for all $(p,a) \in P \times A$ and $v \in (0, \psi(a,p,0))$,

$$\phi_{12}(p,a,v) \leq 0 \text{ and } \phi_{13}(p,a,v) \leq 0$$

5.1 Effects Governing Monotone Matching

- standard productivity complementarity: higher types raise the productivity boost from increase in partner's type ($\phi_{12} > 0$)

- new effect: monotonicity of transferability (type-payoff complementarity)

  - higher partner type a has higher opportunity cost ($\psi_1 > 0$) than $a - \epsilon$ which costs p and p' approximately $|\phi_3 \cdot \psi_1 \cdot \epsilon|$ extra

  - higher types p have flatter frontiers ($\phi_{13} > 0$) than p', hence lower cost of making the transfer

- Conditions in the proposition ensure that both effects go in the same direction: high types have the advantage on both margins and therefore outbid low types

- this is not an ordinal condition: increasing transformations of payoffs may render it invalid (or valid – it is enough for PAM that one representation satisfy condition)

5.2 Example: Principals and Agents

- Problem of interest is

$$\begin{aligned} \phi(p,a,v) = \max\ & R - ps_1 - (1-p)s_0 \\ \text{s.t. } & pV(a+s_1) + (1-p)V(a+s_0) - 1 \geq V(a+s_0), \\ & pV(a+s_1) + (1-p)V(a+s_0) - 1 \geq v \end{aligned}$$

- Let $C(\cdot) \equiv V^{-1}(\cdot)$; both constraints bind, from which

$$\phi(p,a,v) = R + a - pC(\frac{1}{p} + v) - (1-p)C(v).$$

- Thus,

$$\begin{aligned}
\phi_1(a,p,v) &= \frac{1}{p}C'(\frac{1}{p}+v) - C(\frac{1}{p}+v) + C(v)\\
\phi_2(p,a,v) &= 1,\\
\phi_{12} &= 0,\\
\phi_{13}(p,a,v) &= \frac{1}{p}C''(\frac{1}{p}+v) - C'(\frac{1}{p}+v) + C'(v).
\end{aligned}$$

  ϕ_1 is positive since $C'' > 0$, so type monotonicity holds.

- Assume C' is convex (e.g. all utilities of the CRRA class that are weakly more risk averse than square root); then $\phi_{13} \geq 0$

- There is PAM in (p, a) : as long as risk aversion does not decline "too quickly," *agents with lower risk aversion (higher wealth) are matched to principals with projects that are safer, i.e., more likely to succeed*

- The result offers a possible explanation for the finding in Ackerberg-Botticini (2002) that in medieval Tuscany, wealthy peasants were more likely than poor peasants to tend safe crops (cereals) rather than risky ones (vines).

Differential Version of GID

- Recently Chade, Eeckhout and Smith (2017) have provided a condition that is equivalent to GID (necessary as well as sufficient) for differentiable frontiers:

$$\phi_{12}(p, a, v) \geq \frac{-\phi_2(p, a, v)}{\phi_3(p, a, v)} \phi_{13}(p, a, v)$$

- This is derived by recognizing that GID is the same as the Spence-Mirrlees single-crossing condition in (a, v)-space, i.e., that the "marginal rate of substitution" $-\frac{\phi_2(p,a,v)}{\phi_3(p,a,v)}$ is increasing in p.

READINGS

For background reading, see the references in the first paper, especially Becker (1973), Kremer (1993), and Sattinger (1993). The lectures will be based on

Legros, P. and A. Newman (2002), "Monotone Matching in Perfect and Imperfect Worlds," *Review of Economic Studies,* 69: 925-942.

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Ackerberg, D. and M. Botticini (2002): "Endogenous Matching and the Empirical Determinants of Contract Form," *Journal of Political Economy,* 110: 564-591.

Chade, H., J. Eeckhout, and L. Smith (2017), "Sorting through Search and Matching Models in Microeconomics" *Journal of Economic Literature,* 55(2), 493-544.